Description of Distributions of Compound Sums

Chincharauli Zviadi, Makharadze Giorgi
Georgian Technical University

0. Introduction

Many objects in actuarial science are described by the sums of the random variables. Thus, very frequently it is important to analyze the aggregate claims of previous several years, which themselves are the sums of the individual claims.

In contrast to the individual risk model, where the number of summands is deterministic, in the collective risk model this number is a random. So, the situation is more complicated and we have a matter with the dual randomness. Usually, these two variables (the individual claim amount Y and the claim number N) are assumed to be independent. Hence, one of the most important problems is the computation of arising compound distribution. However, even in the case, when N is sufficiently large, the CLT for compound sums may not be accurate by reason that the claims usually have the heavy tails (ex. Pareto, Lognormal etc.). So, the normal approximation does not work and one should evaluate these sums by another way.

One of such ways is the evaluation of the compound distributions recursively: using given recursion for probabilities $p_n = P\{N = n\}$ one tries to obtain the recursion for compound sum S. For the details see the papers [1],[2],[3] in references. This way is very useful for insurance practice but it has some lacks:

1) for some classes of claim number distributions (for example, for general mixed Poisson case, which is very important for insurance practice) one cannot give the probabilities p_n recursively;
2) the recursion for compound sums was derived separately for different classes of recursions for p_n and it is desirable to develop general approach .

On the other hand, it is possible to give the probability generating functions (pgf) for N in general and obtain the form of the pgf of compound distribution as a composition of the pgf of N and (in the case of integer-valued Y) the pgf of Y .Then one can try to find the compound probabilities themselves (calculating the high order derivatives of compound pgf).

In this paper we apply this technique of pgfs for recursive evaluation of the compound distributions and show that by means of such approach one can obtain for example, the Panjer's well-known formulae for compound distributions.

1. Mathematical description of the problem and known results

Let Y_i is the size of the i^{th} claim and N is the number of claims (which we assume to be independent of Y_i's) in a fixed time period, then the aggregate claim is defined as

$$S = \sum_{i=1}^{N} Y_i \ , \tag{1}$$

If we assume that Y_i's are i.i.d. rv's with cumulative distribution function (cdf) F_Y and the distribution of N is given by $p_n = P\{N = n\}$ for all $n \geq 0$, then the cdf of S has the form

$$F_S(x) \equiv P\{S \leq x\} = \sum_{n=0}^{\infty} p_n \cdot F_Y^{n*}(x) \, . \tag{2}$$

So, the computation of cdf of S requires the computation of n-fold convolution of F_Y. However, the explicit expressions for $F_Y^{n*}(x)$ are usually not available and the equation (2) is generally not very useful for calculations of $F_S(x)$.

In order to facilitate the easy evaluation of F_S in equation (2), Panjer (1981) introduced the following family of claim number distributions:

$$p_n = p_{n-1} \cdot (a + b/n), \quad n = 1,2,... \quad , \tag{3}$$

where a and b satisfy only the conditions that

$$\sum_{n=0}^{\infty} p_n = 1 \ , \quad p_n > 0 \, , \quad \forall n \in \mathrm{N} \, . \tag{4}$$

For the probabilities satisfying (3) Panjer (1981) showed that one can calculate the corresponding compound probabilities by the following recursion formulae (we give this formula for arithmetic (i.e. integer valued) Y-s only):

$$g_0 = \sum_{n=0}^{\infty} p_n \cdot f_0^n \ , \qquad g_k = \frac{1}{1 - a \cdot f_0} \cdot \sum_{j=1}^{k} (a + b \cdot j/k) \cdot f_j \cdot g_{k-j} \ , \tag{5}$$

where

$$g_k \equiv P\{S = k\}, \quad f_k \equiv P\{Y = k\}. \tag{6}$$

As we can see, (5) allows us to compute the compound probabilities without using the convolutions.

2. Main Results

The family (3) includes the geometric, Poisson, binomial and negative binomial distributions only. We have a lot of examples in insurance practice, when N does not belong to the distributions listed above. So, more generalizations of recursion of type (3) was needed and this recursion was generalized in many directions by Sundt, Ramsay, Willmot-Panger, Hasselager, Wang-Sobrero (see [4], [5], [6] in references).

Now we introduce the following general form of recursions, which includes all these classes of distributions of N (counting distributions):

$$p_n = (a + b \cdot c_n) \cdot p_{n-1} . \tag{7}$$

Here the sequence c_n can be arbitrary (provided p_n forms a probability distribution). Let us call this recursion as $(a;b;c)$ type recursion.

As we mentioned above, , it is possible using the pgf's technique to obtain more general recursion relations for compound distributions than (5). This method was not applied sufficiently, because of difficulties connected with calculation of high order derivatives of the composed function

$$\varphi_S(t) = \varphi_N\left(\varphi_Y(t)\right), \tag{8}$$

where $\varphi_\xi(x)$ is the pgf of ξ.

For example, for the Panjer's case we have

$$\varphi_S(t) = \left(\frac{1-a}{1-a\cdot\varphi_Y(t)}\right)^{\frac{a+b}{a}} \tag{9}$$

and it is quite difficult to find the high order derivatives with direct differentiation.

This difficulty can be avoided for pgf of general $(a;b;c)$ type recursion as follows. It is easy to obtain the pgf of the distribution given by (7) as

$$\varphi_N(t) = \frac{1-a-b\cdot\int_t^1 \psi(s)ds}{1-at}, \tag{10}$$

where

$$\psi(t) = \sum_{n=0}^{\infty}(n+1)\cdot c_{n+1}\cdot p_n \cdot t^n \tag{11}$$

and so,

$$\varphi_S(t) = \frac{1-a-b\cdot\int_{\varphi_Y(t)}^1 \psi(s)ds}{1-a\cdot\varphi_Y(t)}. \tag{12}$$

Then, by means of (12) for $\varphi_S(t)$ we obtain the equality

$$\left(1-a\varphi_Y(t)\right)\cdot\varphi_S'(t) = \varphi_Y'(t)\cdot[a\varphi_S(t) + b\psi(\varphi_Y(t))], \tag{13}$$

which is the key for solving the problem of calculating the high order derivatives of $\varphi_S(t)$: differentiating both sides of (13) $(k-1)$-times and solving the equation w.r.t. the highest order derivative of $\varphi_S(t)$, we get

$$\varphi_S^{(k)}(t) = \frac{1}{1-a\varphi_Y(t)}\sum_{j=1}^{k} C_k^j \varphi_Y^{(j)}(t)\left(a\varphi_S^{(k-j)}(t) + b\frac{j}{k}[\psi(\varphi_Y(t))]^{(k-j)}\right). \tag{14}$$

As we see from (14), it is possible to express highest order derivatives of $\varphi_S(t)$ with lower order derivatives of the same function. Evidently, for obtaining the compound probabilities, it is sufficient to take $t=0$ in (14) and calculate $\varphi_S^{(k)}(0)/k!$.

Now, focus on (11). It is clear, that the relation between c_n and the function ψ is one to one and so, we can talk about φ as $(a;b;\psi)$ type pgf rather than about probabilities p_n given by $(a;b;c)$ type recursion. If ψ is some function not depending on p_n then by means of (12) one can calculate the derivatives of $\varphi_S(t)$ by direct differentiation. But if c_n is independent from p_n, then function ψ necessarily depends on these probabilities and so, we can assume that it depends on φ_N, and for some known function g, we can write:

$$\psi(t) = g(\varphi_N(t)). \tag{15}$$

Then by means of (8) and (10) the formula for derivatives of $\varphi_S(t)$ takes the following form:

$$\varphi_S^{(k)}(t) = \frac{1}{(1 - a\varphi_Y(t))} \sum_{j=1}^{k} C_k^j \varphi_Y^{(j)}(t) \left(a\varphi_S^{(k-j)}(t) + b\frac{j}{k} [g(\varphi_S(t))]^{(k-j)} \right). \tag{16}$$

For illustration take $g(x) = x$. From (15) it means, that we have $\psi(t) = \varphi_N(t)$ and solving (10) we obtain Panjer's pgf. Finally, from (16) we have

$$\varphi_S^{(k)}(t) = \frac{1}{1 - a\varphi_Y(t)} \cdot \sum_{j=1}^{k} C_k^j \left(a + b\frac{j}{k} \right) \varphi_Y^{(j)}(t) \varphi_S^{(k-j)}(t) \tag{17}$$

and the expression for $\varphi_S^{(k)}(0)/k!$ exactly coincides with (5), where $g_0 = \varphi_S(0) = \varphi(f_0)$.

R e f e r e n c e s

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