ABOUT OPTIMAL SOLUTIONS IN CATASTROPHE RISK INSURANCE

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Problem

Insurance business is generally broken into two classes: life and non-life. One of the reasons of this classification is that usually life insurance is considered to be a long-term business while non-life insurance - as a short-term business. In most cases it's quite true but there are also some exceptions e.g. term life insurance, where term of a policy may not exceed one year and therefore should be considered as a short-term business. In non-life insurance catastrophe insurance is such exception. Although it can be a part of a typical one-year property insurance policy, there are several features that separate catastrophe insurance from other property insurance categories:

1. Catastrophe risks are not independent: a certain disaster can damage all insured property of a region.
2. The probability of a disaster (e.g. earthquake) during one year is relatively smaller than probability of any other claim in an insurer's portfolio.

Distinctions of this kind should enforce regional insurance companies to treat catastrophe insurance as a long-term business unlike other types of property insurance.

It is well-known that in short-term business, one-year total insurance premium should be enough to cover the same year's all claims, as the entire risk of portfolio should be distributed among the insureds and the more independence of risks insured, the better effect of such distribution. It's clear that for a regional insurance company it is impossible to collect a premium in one year that would be enough to cover all claims in case of catastrophe and the reason of this is the dependence among catastrophic risks (if earthquake destroys one house, it is likely that the same will happen to other houses as well). So, a regional insurer can carry out this business itself considering it as a long-term, building up reserves from year to year i.e. distributing the risk in time.

From the point of view of international insurance market catastrophe insurance can be considered to some extent as short-term business. In other words, if regional catastrophic risks are reinsured and there are risks from different regions in reinsurer's portfolio, they can be considered as independent and one-year premium might well be enough to cover the same year's claims. So, alternative way for regional insurer to carry out the business is the reinsurance of the whole risk or some it's part.

Our goal is to create mathematical model for a regional insurance company involved in catastrophe insurance business and find the best strategy of managing such risks. We will use for this Controlled Markov Chain Theory, which seems to be an appropriate tool for solving dynamic decision-making issues.

Basic Model

Consider a regional insurance company with a portfolio of catastrophe risks. We assume that the present value of overall cost of the insured property M and of the total annual premiums S do not change throughout the years. Whole premium is collected in the beginning of a year and all claims of the same year are covered at the end of it.

We also assume that during every year can happen a disaster (earthquake) causing total amount of claims M_1 $(M_1 < M)$ after which the insured property will be restorable or a "complete disaster" with amount of claims M that completely destroys the insured property and causes termination of the portfolio. Of course there's a probability that no disaster will happen during a year, thus probability distribution of overall claims of any year is:

$$\xi_i \sim \begin{pmatrix} 0 & M_1 & M \\ 1-\theta & p\theta & (1-p)\theta \end{pmatrix}$$

where θ is a probability of a disaster during one year and p a probability that the property will be restorable after it.

In the first model we consider a situation where a company has just two diametral options: either reinsure whole portfolio or keep it entirely with itself.

Our goal is to find best long-term strategy for a regional company in such circumstances.

Mathematical model of the situation described could be a model based on the Controlled Markov Chain Theory, which is given below:

Countable space of states $X = \{1,2,3,...,k,...\}$ where each state represents a certain year;

A two-element space of decisions $A = \{0,1\}$ where 0 denotes a decision to take no reinsurance and 1 a decision to reinsure completely.

Break moment τ independent of the process itself, with conditional probability of break equal to $\delta = (1-p)\theta$ (probability of "complete disaster") at every step under condition that there were no breaks at previous steps.

The problem of finding optimal solution can be stated as:

Find the maximum of the mean of some additive criteria L depended on trajectories of the Controlled Markov Chain and the break moment τ on the space of stationary strategies $F = \{f_1, f_2, f_3, ...\}$, with:

$$f_i = (a_1, a_2, a_3, ...), \quad a_i = \begin{cases} 1 \\ 0 \end{cases}$$

It is well known from the classical theory [1;2] that optimal strategy in the class of stationary strategies is also optimal in the class of general strategies.

Additive criteria in our case will be:

$$L = \sum_{n=1}^{\tau-1} r(x_n, a_n) + c(x_\tau, a_\tau)$$

Functions r and c determined on $X \times A$ are given:

$$r(j,1) = \alpha S$$

$$r(j,0) = S - \frac{\theta p}{1-\theta+\theta p} M_1$$

$$c(j,1) = \alpha S$$

$$c(j,0) = S - M \ ,$$

where α is a part of the total premium left after paying a reinsurance premium(e.g. a reinsurance commissions). $r(j,1)$ is a cash flow of the year j if the portfolio was reinsured and $r(j,0)$ is a cash flow of the same year if no reinsurance was taken. $c(j,1)$ and $c(j,0)$ are cash flows of the year in which "complete disaster" took place(last year of the business).

The mean of the criteria L for given $f \in F$ will be:

$$E^f L = R^f = \sum_{n=1}^{\infty} (1-\delta)^{n-1} \left[(1-\delta) r^f(n) + \delta c^f(n)\right]$$

where $r^f(n)$ and $c^f(n)$ are nth elements of vectors $r^f = (r(1, f(1)), r(2, f(2)), ...)^T$ and $c^f = c(c(1, f(1)), c(2, f(2)), ...)^T$.

The next theorem stands for the problem stated above:

Theorem 1. Consider strategies

$$f_0 = (a_1, a_2, ..., 0, a_{m+1}, ...) \quad \text{and} \quad f_1 = (a_1, a_2, ..., 1, a_{m+1}, ...)$$

if

$$(1-\alpha)S \le \left(\theta p M_1 + \delta M\right) \qquad (1)$$

then

$$R^{f_1} \ge R^{f_0} .$$

And if

$$(1-\alpha)S \ge \left(\theta p M_1 + \delta M\right) \qquad (2)$$

then

$$R^{f_1} \le R^{f_0} .$$

The consequence of the theorem is that under the condition (1) optimal strategy is $f^* = (1,1,1,...)$ and under the condition (2) optimal strategy is $f^{**} = (0,0,0,...)$. Taking a closer look at the conditions (1) and (2) one will find out that they compare reinsurance premium with average annual portfolio risk. It should be mentioned, that in practice condition (1) holds as a rule (due to the distribution of the risk over the world) and therefore for regional insurance companies the best solution is to reinsure their catastrophic risks every year.

The model of course is subject to some generalizations and one of them is considering different types of decision spaces.

Theorem 2. Consider a space of decisions $A = \{q_1, q_2, ..., q_n\}$, where q_i $(q_i < q_{i+1}, i = 1,...,n)$ denotes a decision to take a Quota Share reinsurance with retention $(1 - q_i)$. If $(1-\alpha)S \le (\theta p M_1 + \delta M)$, then the optimal strategy is $f^* = (q_1, q_1, ..., q_1)$, and if $(1-\alpha)S \ge (\theta p M_1 + \delta M)$, then the optimal strategy is $f^{**} = (q_n, q_n, ..., q_n)$.

Theorem 3. Consider a space of decisions $A = \{0,1\}$, where 1 denotes a decision to take a Stop Loss reinsurance cover with retention level equal to M_1 and 0 denotes a decision to take no reinsurance. If $(1-\alpha)S \le (1-p)\theta(M - M_1)$, then the optimal strategy is $f^* = (1,1,1,...)$, and if $(1-\alpha)S \ge (1-p)\theta(M - M_1)$, then the optimal strategy is $f^{**} = (0,0,0,...)$.

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